ON MEAN-SQUARE OPTIMUM STABILIZATION AT DAMPED RANDOM PERTURBATIONS

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This paper deals with the problem of design of a stabilizing control in a linear system with damped random perturbations. This control is determined by the condition of minimum of the mathematical expectation of the integral squared deviation. The investigation generalizes a result of Letov [1]. The problem is solved by the use of Liapunov functions [3,4], modified according to Bellman's principles of dynamic programming [4]. This approach to the problems of optimum control, based on the concept of optimum Liapunov functions, is described in [5,6]. The author draws attention to the fact that during the writing of this paper he was familiar with the investigations of M.E. Salukvadze on the mean-square optimum stabilization for stationary perturbations.

1. Formulation of the problem. We shall consider a control system described by the equations

$$\frac{dx_i}{dt} = a_{i1}x_1 + \ldots + a_{in}x_n + m_i\xi + \varphi_i(t,\eta) \quad (i = 1, \ldots, n) \qquad (1.1)$$

where x_i are the deviations of the n-dimensional vectorial controlled quantity x from its given value $x_i = 0, (i = 1, \ldots, n)$, ξ is the stabilizing input of the control, and $\phi_i(t, \eta)$ are random disturbances. The quantities $\phi_i(t, \eta)$ are considered to be functions of time t and a random r-dimensional variable $\eta(t)$. In a particular case it may be that $n = r$, and the components of the vector ϕ may coincide with the components of $\eta(t)$. We assume that the random function $\eta(t)$ describes a stochastic Markovian process with a known probabilistic transition function [7, pp. 232-247]

$$p[s, \alpha; t, B] = P[\eta(t) \in B / \eta(s) = \alpha]$$

admitting the decomposition

$$p\ [s, \alpha; t, \{\alpha\}] = 1 - q\ (t, \alpha)\ (t - s) + o\ (t - s) \tag{1.2}$$

$$p\ [s, \alpha; t, B] = q\ (t, \alpha, B)\ (t - s) + o\ (t - s) \tag{1.3}$$

(α does not belong to B)

Here, α and B are r-dimensional vectors and r-dimensional Borelian sets, $s < t$ are instants of time, the symbol $P[\ Q/L\ ]$ denotes the probability of the event Q at the condition L, and $p(\Delta t)$ is an infinitesimal quantity of a higher order than Δt.

This paper deals with the problem of design of a control action (or, shortly, control) which assures that the perturbed motion described by (1.1) approaches asymptotically the state $x = 0$ (for $t \to \infty$). Therefore, we limit ourselves to the case of the disturbances $\phi_i(t, \eta)$ which decrease sufficiently fast as time increases. (If the disturbances $\phi_i(t, \eta)$ do not vanish for $t \to \infty$, then the approach of the motion $x(t)$, given by (1.1), to the point $x = 0$ is possible only within a certain unreducible error $\delta > 0$.)

We denote by the symbol $M[\zeta/L\ ]$ the mathematical expectation of the random quantity ζ at the condition L. The disturbances ϕ_i are said to be bounded and decreasing in the mean if a function $f[\ t_0,\ t\ ]$ may be found, determined and continuous for $0 \leqslant t_0 \leqslant t$, such that the following conditions are satisfied:

$$|\, f\ [t_0, t]\, | \leqslant N = \text{const} \tag{1.4}$$

$$M[|\varphi_k\ (t, \eta)|\ /\ \eta\ (t_0)\ \text{—arbitrary}] \leqslant f\ [t_0, t]\ \ (t \geqslant t_0) \tag{1.5}$$

$$f\ (t_0) = \int_{t_0}^{\infty} f\ [t_0, t]\, dt < \infty \tag{1.6}$$

$$\int_0^{\infty} f^2\ (t)\, dt < \infty, \qquad \lim f\ (t) = 0 \quad \text{for } t \to \infty \tag{1.7}$$

Note 1.1. In view of the conditions (1.4) and (1.5), the remark that ϕ may be equal to η (see above, p. 1212) requires a qualification. In this case the variable η may assume only the values in the region $\|\eta\| < N$, and the symbols α and B in (1.2) and (1.3) denote then the vectors and subsets in the region $\|\eta\| < N$, respectively. (Here and in the following, the symbol $\|y\|$ denotes the Euclidean absolute value of the vector y, i.e.

$$\|y\| = (y_1^2 + \ldots + y_n^2)^{1/2})$$

The control ξ will be sought in the form of a function $\xi = \xi[x, \eta, t]$. This corresponds to the possibility of measuring the quantities $x_i(t)$ and $\eta_k(t)$ from the signals entering the control system during the process of control [8].

If the function $\xi[x, \eta, t]$ is selected, then arbitrary initial conditions x_0, η_0 for $t = t_0$, generate a Markovian stochastic process for $t \geqslant t_0$ in the space $\{x, \eta\}$ [7, p.72], on the basis of Equations (1.1).

We shall not define here rigorously the concept of solution of stochastic equations (1.1) for a general case. We shall limit ourselves to simple cases where the stochastic solution $x(t)$ of Equations (1.1) may be determined without difficulties.

We shall assume namely that the functions $\phi_i(t, \eta)$ are continuous in both variables and the realizations $\eta^{(p)}(t)$ of random functions $\eta(t)$ are piecewise continuous functions. (The conditions under which the realizations $\eta^{(p)}(t)$ actually have this property are shown, for instance, in [7, p.242].)

With this assumption, as realizations of a stochastic solution $\{x^{(p)}(t), \eta^{(p)}(t)\}$ of the system (1.1) we shall consider, together with the realizations $\eta^{(p)}(t)$, the continuous functions $x^{(p)}(t)$ satisfying Equations (1.1) in the intervals of constant $\eta^{(p)}(t)$. The Markov vector-function $\{x(t), \eta(t)\}$ generated by the initial conditions x_0, η_0 for $t = t_0$ by virtue of Equations (1.1) with $\xi = \zeta$ will be denoted by the symbol

$$\{x(t), \eta(t) / x_0, \eta_0, t_0; \zeta\}$$

The problem consists of the following. It is necessary to determine the optimum control ξ°, i.e. the function $\xi^\circ[x, \eta, t]$ satisfying the conditions:

Condition 1.1. Every realization $x^{(p)}(t)$ of the solution

$$\{x(t), \eta(t) / x_0, \eta_0, t_0; \xi^\circ\}$$

should be bounded in its absolute value by a constant depending on the initial conditions $x_0 \in (-\infty, \infty)$, η_0, $t_0 \geqslant 0$, i.e. for $t_0 \leqslant t < \infty$

$$\| x^{(p)}(t) \| \leqslant N(x_0, \eta_0, t_0) \tag{1.8}$$

Condition 1.2. For an arbitrary initial condition $\{x_0, \eta_0, t_0\}$, the solution $\{x(t), \eta(t)/x_0, \eta_0, t_0, \xi^\circ\}$ should asymptotically approach in mean-square the point $x = 0$ for $t \to \infty$, i.e.

$$\lim M[\| x(t) \|^2 / x_0, \eta_0, t_0; \xi^\circ] = 0 \quad \text{for } t \to \infty \tag{1.9}$$

Condition 1.3. For an arbitrary initial condition $\{x_0, \eta_0, t_0\}$ the quantity

$$J\,[x_0, \eta_0, t_0;\ \xi] = \int_{t_0}^{\infty} \mathrm{M}\Big[\Big(\sum_{i=1}^{n} x_i^2\,(t) + \xi^2\,(t)\Big) \,/\, x_0, \eta_0, t_0;\ \xi\Big] dt \qquad (1.10)$$

should be finite for $\xi = \xi^\circ$ and should be minimum for this control taken from a family of functions $\{\xi\}$ specified in advance. As an admissible class of functions $\{\xi\}$ such functions $\{\xi\,[x, \eta, t]\}$ will be selected for which the solutions of Equations (1.1) may be determined in the way described above (p. 1214), and the use can be made of the generalized derivative of the Liapunov function, which will be introduced later. (A rigorous discussion of this problem would distract us from the principal task, i.e. the construction of optimum ξ°. In any case, with known regularity of the functions $p[s, a;\ t, B]$ and $\phi_k(t, \eta)$ we may limit ourselves to continuous admissible functions $\{\xi\,[x, \eta, t]\}$, as it follows from the form of the solution ξ°.)

2. The method of solution of the problem. Let the functions $v(x, \eta, t)$ and $\xi^\circ[x, \eta, t]$ be found satisfying the conditions:

Condition 2.1. The decomposition

$$v\,(x, \eta, t) = v_2\,(x) + v_1\,(x, \eta, t) + v_0\,(\eta, t) \qquad (2.1)$$

is valid, where

$$v_2\,(x) = \sum_{i,j=1}^{n} b_{ij} x_i x_j, \qquad b_{ij} = \mathrm{const} \qquad (2.2)$$

is a strongly positive-definite quadratic form, the function

$$v_1\,(x, \eta, t) = \sum_{i=1}^{n} b_i\,(\eta, t)\, x_i \qquad (2.3)$$

is a linear form whose coefficients $b_i(\eta, t)$ satisfy the inequalities

$$|\, b_i\,(\eta, t)\,| \leqslant N_1 = \mathrm{const} \qquad (i = 1, \ldots, n) \qquad (2.4)$$

and the limit relations

$$\lim\,[b_i\,(\eta, t)] = 0 \quad \text{for } t \to \infty \qquad (t = 1, \ldots, n) \qquad (2.5)$$

and the function $v_0(\eta, t)$ satisfies the inequality

$$|\, v_0\,(\eta, t)\,| \leqslant N_0 = \mathrm{const} \qquad (2.6)$$

and the limit relation

$$\lim \boldsymbol{M}\,[|\,v_0(\eta, t)\,|\,/\,\eta_0, t_0] = 0 \quad \text{for } t \to \infty \tag{2.7}$$

for arbitrary initial conditions η_0.

Condition 2.2. The generalized derivative [6] of the function v with respect to Equations (1.1), for $\xi = \xi^\circ$ $(d\boldsymbol{M}\{v\}/dt)_{\xi^\circ}$ satisfies the equations

$$\left(\frac{d\boldsymbol{M}\{v\}}{dt}\right)_{\xi_0} = -\sum_{i=1}^{n} x_i^2 - (\xi^\circ)^2 \tag{2.8}$$

$$\min\left[\left(\frac{d\boldsymbol{M}\{v\}}{dt}\right)_{\xi} + \sum_{i=1}^{n} x_i^2 + \xi^2\right] = \left(\frac{d\boldsymbol{M}\{v\}}{dt}\right)_{\xi^\circ} + \sum_{i=1}^{n} x_i^2 + (\xi^\circ)^2 \tag{2.9}$$

for all the values of x, η, and t.

Condition 2.3. The generalized derivatives $(d\boldsymbol{M}\{v_1\}/dt)_{\xi^\circ}$ and $d\boldsymbol{M}\{v_0\}/dt$ satisfy the inequalities

$$\left|\left(\frac{d\boldsymbol{M}\{v_1\}}{dt}\right)_{\xi^\circ}\right| \leqslant N_1\|x\|, \qquad \left|\frac{d\boldsymbol{M}\{v_0\}}{dt}\right| \leqslant N_0 \tag{2.10}$$

Thus, ξ° is the optimum control and

$$v(x_0, \eta_0, t_0) = J\,[x_0, \eta_0, t_0; \xi^\circ] \tag{2.11}$$

We shall prove this proposition. That the condition 1.1 is satisfied may be shown in the following way.

The derivative $(dv_2/dt)^{(p)}_{(1.1)}$ of the function $v_2(x)$ with respect to Equations (1.1), for an arbitrary realization $\eta^{(p)}(t)$ and for sufficiently large values of the norm $\|x\|$, will be a strongly-definite negative function. As a result of the conditions (2.10) and of the fact that the functions ϕ_i are bounded (the conditions (1.4) and (1.5), the derivatives $(d\boldsymbol{M}\{v\}/dt)_{\xi^\circ}$ and $(dv_2/dt)^{(p)}_{(1.1)}$ differ only by some bounded functions of time t and the random variable η, and also by some functions of t, η, and x which do not increase faster than $\|x\|$ with increasing $\|x\|$. Thus, our statement that $(dv_2/dt)^{(p)}_{(1.1)}$ is negative follows from the condition (2.8). Now, it can be shown that the realizations $x^{(p)}(t)$ are bounded by the use of the function $v_2(x)$ and standard arguments of the theory of stability.

Let us check whether condition 1.2 is satisfied. Consider the random function (solution)

$$\{x(t), \eta(t) / x_0, \eta_0, t_0; \xi^\circ\}$$

We construct the quantity

$$V_t[\xi^\circ] = \boldsymbol{M}[v(x(t), \eta(t), t) / x_0, \eta_0, t_0; \xi^\circ] \tag{2.12}$$

and calculate its derivative with respect to time t at the instant $t = \tau$. Considering the Markovian property of the vector-function $\{x(t), \eta(t)\}$, we can write

$$\left(\frac{dV_t[\xi^\circ]}{dt}\right)_{t=\tau} = \boldsymbol{M}\left[\left(\frac{d\boldsymbol{M}\{v(x(t), \eta(t), t) / x(\tau), \eta(\tau), \tau; \xi^\circ\}}{dt}\right)_{t=\tau} \Big/ x_0, \eta_0, t_0; \xi^\circ\right]$$
$$= -\boldsymbol{M}[\|x(\tau)\|^2 + (\xi^\circ(\tau))^2 / x_0, \eta_0, t_0; \xi^\circ] \tag{2.13}$$

Integrating Equation (2.13) with respect to τ from $\tau = t_0$ to $\tau = T > t_0$, we obtain the equation

$$V_T[\xi^\circ] - v(x_0, \eta_0, t_0) = -\int_{t_0}^{T} \boldsymbol{M}[\|x(\tau)\|^2 + (\xi^\circ(\tau))^2 / x_0, \eta_0, t_0; \xi^\circ]\, d\tau \tag{2.14}$$

For $T \to \infty$ the quantity $V_T[\xi^\circ]$ converges to the quantity $\boldsymbol{M}[v_2(x(T)) / x_0, \eta_0, t_0; \xi^\circ]$ because of the conditions (2.5) and (2.7) and the realization $x^{(p)}(t)$ being bounded (see the condition 1.1, whose satisfaction has already been shown). This and the fact that the function $v_2(t)$ is strongly positive-definite, imply that the lower bound of the first term on the left-hand side of Equation (2.14) is non-negative for $T \to \infty$. Therefore, the integral on the right-hand side of Equation (2.14) converges for $T \to \infty$. From the convergence of this integral we conclude that

$$\overline{\lim_{\tau\to\infty}}\, \boldsymbol{M}[\|x(\tau)\|^2 / x_0, \eta_0, t_0; \quad \xi^\circ] = 0$$

Since the quadratic form $v_2(x)$ satisfies the inequality

$$v_2(x) \leqslant \lambda \|x\|^2 \quad (\lambda = \text{const})$$

and the equation

$$\lim_{\tau\to\infty} \boldsymbol{M}[v_2(x(\tau)) / x_0, \eta_0, t_0; \xi^\circ] = \lim_{\tau\to\infty} \boldsymbol{M}[v(x(\tau), \eta(\tau), \tau) / x_0, \eta_0, t_0; \xi^\circ]$$

we finally establish the validity of the limit relation

$$\lim V_T[\xi^\circ] = 0 \quad \text{for } T \to \infty$$

The existence of this limit follows from the monotonicy of $V_t[\xi^\circ]$ in t $(dV_t/dt \leqslant 0)$.

Consequently, the equation

$$v(x_0, \eta_0, t_0) = \int_{t_0}^{\infty} M\,[\|x(\tau)\|^2 + (\xi^\circ(\tau))^2 / x_0, \eta_0, t_0; \xi^\circ]\, d\tau$$

is valid, which proves that (2.11) is satisfied.

We note that the limit relations derived here imply also satisfaction of the condition 1.2, since the form $v_2(x)$ is strongly positive-definite and, thus, the inequality holds $v_2(x) \geqslant \epsilon \|x\|^2$ $(\epsilon > 0,$ const).

The satisfaction of the condition 1.3 remains to be verified. Suppose that this condition is not satisfied and, consequently, an optimal control ξ^* different from ξ° exists, and for an arbitrary initial condition $\{x_0, \eta_0, t_0\}$ the control ξ^* yields the inequality

$$J\,[x_0, \eta_0, t_0; \xi^*] < J\,[x_0, \eta_0, t_0; \xi^\circ] \tag{2.15}$$

We shall show that this assumption leads to a contradiction. From the condition (2.9) we conclude that

$$\left(\frac{dM\{v\}}{dt}\right)_{\xi^*} \geqslant -\|x^2\| - (\xi^*)^2 \tag{2.16}$$

Introducing the quantity $V_t[\xi^*]$ analogous to the previous one and integrating the inequality (2.16) similarly to what was done in the case of Equation (2.8), we obtain the inequality

$$V_T[\xi^*] - v(x_0, \eta_0, t_0) \geqslant -\int_{t_0}^{\infty} M\,[\|x(\tau)\|^2 + (\xi^*(\tau))^2 / x_0, \eta_0, t_0; \xi^*]\, d\tau \tag{2.17}$$

Since it is assumed that ξ^* is an optimum control, the norm $\|x^{(p)}(t)\|$ of the realization of the solution $\{x(t), \eta(t)/x_0, \eta_0, t_0; \xi^*\}$ should be bounded. Consequently, as before, the validity of the limit relation

$$\lim V_T\,[\xi^*] = \lim M\,[v_2(x(T)) \,/\, x_0, \eta_0, t_0; \xi^*]$$

may be proved for $T \to \infty$, if the limit on the right-hand side exists. From this relation, from the condition 1.2, and from the inequality $v_2(x) \leqslant \lambda \|x\|^2$, we conclude that for $T \to \infty$ the first term on the left-hand side of (2.17) converges to zero. According to the condition 1.3, the integral on the right-hand side of (2.17) should converge. Therefore, from (2.17) by a limiting procedure for $T \to \infty$ we derive the inequality

$$\int_{t_0}^{\infty} M\,[\|x(\tau)\|^2 + (\xi(\tau)^*)^2 / x_0, \eta_0, t_0; \xi^*]\, d\tau \geqslant v(x_0, \eta_0, t_0)$$

i.e.

$$J\,[x_0, \eta_0, t_0; \xi^*] \geqslant v\,(x_0, \eta_0, t_0)$$

The last inequality is contradictory to our assumption (2.15) and Equation (2.11). This implies that the condition 1.3) is satisfied for ξ°.

Thus, the problem of construction of the optimum control $\xi^\circ[x, \eta, t]$ reduces to determination of the functions v and ξ° satisfying the conditions 2.1 to 2.3.

3. Construction of the optimum control ξ°. In this section the conditions of solvability of the problem are clarified, and the form of the optimum control ξ° is established. We shall write Equation (2.8) for the function v in explicit form, which will be called the optimum Liapunov function.

For this purpose, it is necessary to know the expression for the derivative $(d\,M\{v\}/dt)_\xi$ in terms of the parameters of the system (1.1) and the stochastic characteristics of the random function $\eta(t)$. It is

$$\frac{dM\{v\}}{dt} = \frac{dM\{v_2\}}{dt} + \frac{dM\{v_1\}}{dt} + \frac{dM\{v_0\}}{dt}$$

Let us calculate the derivative $d\,M\{v_1\}/dt$. This calculation will be performed analogously to the one in [6]. We have

$$dM\{v_1\}/dt = \lim\,[\Delta M\,\{v_1\}\,/\,\Delta t] \quad \text{for } \Delta t \to +0$$

We determine $\Delta\,M\{v_1\}$. Neglecting infinitesimal quantities of higher order with respect to Δt, we consider that in the interval Δt two mutually exclusive effects may occur:

1) the event D: the quantity η maintains its value, i.e.

$$\eta\,(t + \Delta t) = \eta\,(t) = \alpha$$

2) the event D^{-1}: the quantity η changes its value once.

According to (1.2), the probability of the event D is $P[D] \approx 1 - q(\eta,\ t)\Delta t$, and the probability of the opposite event is $P[D^{-1}] \approx q(\eta,\ t)\Delta t$.

The event D^{-1} may be split into a finite number of events $D_1^{-1}, \ldots, D_m^{-1}$ where an event D_k^{-1}, $k < m$, is a single change of the quantity $\eta(\tau)$ in the interval $t < \tau \leqslant t + \Delta t$, i.e. $\eta(t + \Delta t) \neq \eta(t)$ with $\eta(t + \Delta t) \in B_k = (\beta_{k-1}, \beta_k)$ (or $\eta(t + \Delta t) \in B_m = (\beta_{m-1}, \beta_m)$ for $k = m$) and

$\{\beta_k\}$ is a sequence of increasing numbers, $\beta_0 = -\infty$, $\beta_m = \infty$.

(We note that here the case of a scalar variable has been considered; in a general case the considerations are analogous.)

According to (1.3) the probability is

$$P\,[D_k^{-1}] \approx q\,(t, \alpha, B_k)\,\Delta t$$

If the distribution function $\gamma(t, \alpha, \beta) = q(t, \alpha, B(\beta))$ is introduced, with $B(\beta)$ being the semi-interval $(-\infty, \beta)$ without α, then

$$P\,[D_k^{-1}] \approx [\gamma\,(t, \alpha, \beta_k) - \gamma\,(t, \alpha, \beta_{k-1})]\,\Delta t$$

We have the equality

$$\Delta M\,\{v_1\} = \Delta_D v_1 P\,[D] + \sum_{k=1}^{m} \Delta_k v_1 P\,[D_k^{-1}] \tag{3.1}$$

where $\Delta_D v_1$ is the change of v_1 in the case of the event D, and $\Delta_k v_1$ is the change of v_1 in the case of the event D_k^{-1}. In the case of the event D, Equations (1.1) may be considered in the interval Δt as ordinary differential equations, and $\Delta_D v_1$ may be determined using the formula of finite increments as is done in a classical case. For the realization D_k^{-1} we assume $\Delta_k v_1 \approx v_1(x, \beta, t) - v_1(x, \eta, t)$ where $\beta \in B_k$. Substituting the probabilities $P[\,D\,]$ and $P[\,D_k^{-1}\,]$ into the equality (3.1), and passing to the limit for $\Delta t \to 0$ while increasing to infinity the number m of divisions β_k

$$(\beta_k - \beta_{k-1} \to 0\ (k = 2, \ldots, m-1))$$

we obtain at the point (x, η, t)

$$\left(\frac{dM\,\{v_1\}}{dt}\right)_{(x,\eta,t)} = \sum_{i=1}^{n} \frac{\partial v_1\,(x, \eta, t)}{\partial x_i}\left[\sum_{j=1}^{n} a_{ij}x_j + m_i\xi\,[x, \eta, t] + \varphi_i\,(t, \eta)\right] +$$
$$+ \frac{\partial v_1}{\partial t} + \int_{-\infty}^{\infty} v_1\,(x, \beta, t)\,d_\beta\,\gamma\,(t, \eta, \beta) - v_1\,(x, \eta, t)\,q\,(t, \eta) \tag{3.2}$$

Here, the integral is taken in the sense of Stieltjes, and the symbol $q(t, \eta)$ denotes the quantity $\gamma(t, \eta, \infty)$.

The derivative $d\boldsymbol{M}\{v_2\}/dt$ reduces simply to the derivative $(dv_2/dt)_{(1.1)}$ (for $\eta = \text{const}$), since v_2 does not depend explicitly on η.

Therefore

$$\left(\frac{dM\{v_2\}}{dt}\right)_{(x,\eta,t)} = \sum_{i=1}^{n} \frac{\partial v_2(x)}{\partial x_i}\left[\sum_{i=1}^{n} a_{ij}x_j + m_i\xi\,[x,\eta,t] + \varphi_i(t,\eta)\right] \tag{3.3}$$

Considering (3.1), (3.2), and (3.3), Equation (2.8) may be written in the explicit form

$$\sum_{i=1}^{n}\left(\frac{\partial v_2(x)}{\partial x_i} + \frac{\partial v_1(x,\eta,t)}{\partial x_i}\right)\left[\sum_{i=1}^{n} a_{ij}x_j + m_i\xi\,[x,\eta,t] + \varphi_i(t,\eta)\right] + \frac{\partial v_1}{\partial t} +$$
$$+ \int_{-\infty}^{\infty} v_1(x,\beta,t)\, d_\beta\, \gamma(t,\eta,\beta) - v_1(x,\eta,t)\, q(t,\eta) +$$
$$+ \frac{dM\{v_0(\eta,t)\}}{dt} + \sum_{i=1}^{n} x_i^2 + \xi^2\,[x,\eta,t) = 0 \tag{3.4}$$

We shall now set up the equations corresponding to the condition (2.9). According to this condition the left-hand side of Equation (3.4) should be minimum for $\xi = \xi^\circ$. Thus, the second equation for v and ξ° is obtained from (3.4) by differentiating (varying) this equality with respect to ξ.

We have

$$\sum_{i=1}^{n}\left[\frac{\partial v_2(x)}{\partial x_i} + \frac{\partial v_1(x,\eta,t)}{\partial x_i}\right] m_i + 2\xi\,[x,\eta,t] = 0 \tag{3.5}$$

Let us discuss these equations. We assume that Equations (3.4) and (3.5) have solutions v_2, v_1, and v_0 of the form described above in the conditions 2.1. Thus, from (3.5) it follows that the optimum control

$$\xi^\circ\,[x,\eta,t] = -\frac{1}{2}\left(\sum_{i=1}^{n}\left[\frac{\partial v_2(x)}{\partial x_i} + \frac{\partial v_1(x,\eta,t)}{\partial x_i}\right] m_i\right) \tag{3.6}$$

consists of the linear function

$$\xi_1^\circ = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial v_2(x)}{\partial x_i} m_i = \sum_{j=1}^{n}\mu_j x_j \qquad (\mu_j = \mathrm{const}) \tag{3.7}$$

and the random term in the form

$$\xi_*^\circ = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial v_1(x,\eta,t)}{\partial x_i} m_i = -\frac{1}{2}\sum_{i=1}^{n} b_i(\eta,t)\, m_i \tag{3.8}$$

We shall show that under certain known conditions the problem can actually be solved in the form (3.6). We substitute Expression (3.6) for ξ° into Equation (3.4). The equation obtained may be satisfied by making equal the terms with the same power of x_i. We write the equations obtained in this manner:

$$(x^2) \qquad \sum_{i=1}^{n} \frac{\partial v^2(x)}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}x_j\right) - \frac{1}{4}\left(\sum_{i=1}^{n} \frac{\partial v_2(x)}{\partial x_i} m_i\right)^2 = -\sum_{i=1}^{n} x_i^2 \tag{3.9}$$

$$(x^1) \quad \sum_{i=1}^{n} \frac{\partial v_1(x, \eta, t)}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}x_j - \frac{1}{2} m_i \sum_{j=1}^{n} \frac{\partial v_2(x)}{\partial x_j} m_j\right) + \frac{\partial v_1}{\partial t} + \tag{3.10}$$

$$+ \int_{-\infty}^{\infty} v_1(x, \eta, t)\, d_\beta \gamma(\eta, \beta, t) - q(t, \eta)\, v_1(x, \eta, t) = -\sum_{i=1}^{n} \frac{\partial v_2(x)}{\partial x_i} \varphi_i(t, \eta)$$

$$(x^0) \quad \frac{dM\{v_0\}}{dt} = -\sum_{i=1}^{n} \frac{\partial v_1(x, \eta, t)}{\partial x_i} \varphi_i(\eta, t) + \frac{1}{4}\left(\sum_{i=1}^{n} \frac{\partial v_1(x, \eta, t)}{\partial x_i} m_i\right)^2 \tag{3.11}$$

Let us consider Equation (3.9) This equation has the same form as the equation for the optimum Liapunov function, derived in [1] discussing the problem of construction of the optimum control ξ° in the absence of the perturbations $\phi_i(t, \eta)$. The necessary and sufficient conditions of solvability of this equation in the form of a positive-definite quadratic form have been established by Kirillova [9]. We shall give here these conditions in order to make our presentation complete. Equation (3.9) has a solution in the form of a positive-definite quadratic form if, and only if, the following condition is satisfied.

Condition 3.1. The linear subspace of an n-dimensional vector space defined by the vectors $m, Am, \ldots, A^{n-1}m$ contains the total subspace of the matrix A corresponding to the eigenvalues λ_k with non-negative real parts. Here A is the matrix of the coefficients a_{ij}, and m is the vector $\{m_i\}$.

In particular, a sufficient condition of solvability of Equation (3.9) in the form of a positive-definite quadratic form $v_2(x)$ is linear independence of the vectors $m, Am, \ldots, A^{n-1}m$. In the following we shall assume that the condition 3.1 is satisfied.

Let the function $v_2(x)$ be found from Equation (3.9). Thus, the linear part ξ_1° of the optimum control is determined from the relation (3.7). Substituting ξ_1°, instead of ξ, into Equation (1.1), we obtain an auxiliary system of equations

$$\frac{dx_i}{dt} = a_{i1}x_1 + \ldots + a_{in}x_n - \frac{1}{2} \quad {}_i\sum_{j=1}^{n} \frac{\partial v_2(x)}{\partial x_j} m_j \quad (i = 1, \ldots, n) \tag{3.12}$$

The solutions of the system (3.12) are obviously asymptotically stable, since the function $v_2(x)$ is positive-definite and its derivative $(dv_2/dt)_{(3.12)}$ on the strength of the system of equations (3.12) is negative-definite, i.e. all the assumptions of the Liapunov theorem of asymptotic stability are satisfied [2, p.90].

Let us consider now Equation (3.10). It is convenient to discuss this equation in the following way.

On the left-hand side of Equation (3.10) is a quantity equal to the generalized derivative $(d\,M\{v_1\}/dt)_{(3.12)}$ of the function v_1 with respect to Equations (3.12), i.e. we have

$$\left(\frac{dM\{v_1\}}{dt}\right)_{(3.12)} = -\sum_{i=1}^{n} \frac{\partial v_2(x)}{\partial x_i} \varphi_i(t, \eta) \tag{3.13}$$

With this interpretation of Equation (3.10) it is easy to show that this equation has a solution in the form of a linear form (2.3), satisfying the conditions (2.4) and (2.5). In fact, the form $v_1(x, \eta, t)$ can be determined by the formula

$$v_1(x, \eta, t) = \int_t^\infty M\left[\sum_{i=1}^{n} \frac{\partial v_2(x(\tau))}{\partial x_i} \varphi_i(\tau, \eta(\tau))/x, \eta, t;\ (3.12)\right] d\tau \tag{3.14}$$

where (3.12) in the symbol for mathematical expectation M under the integral sign in (3.14) indicates that $x(\tau)$ is the solution of the system (3.12) corresponding to the initial condition x for $\tau = t$. We shall now verify whether the conditions (2.3), (2.4), and (2.5) are satisfied by the function v_1 determined by Equation (3.14). In order to establish the linearity of the function v_1 (3.14), we write this expression out in full. If $F(t)$ denotes the fundamental matrix of solutions of the system (3.12) ($F(0) = E$ being the unitary matrix) then

$$\frac{\partial v_2(x(\tau))}{\partial x_i} = 2\sum_{j=1}^{n} b_{ij}\{F(\tau - t)x\}_j .$$

where $\{F(\tau - t)x\}_j$ is the jth row of the product of the matrix $F(\tau - t)$ and the vector x. Now Equation (3.14) can be written as

$$v_1(x, \eta, t) = 2\int_t^\infty \left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}\{F(\tau - t)x\}_j\right) M[\varphi_i(\tau, \eta(\tau))/\eta(t), t]\right\} d\tau \tag{3.15}$$

Since $F(t)$ is the fundamental matrix of solutions of the asymptotically stable system (3.12), the elements of this matrix should approach zero for increasing time, and

$$\|F(t)\| \leqslant N_F e^{-\alpha t} \qquad (t \geqslant 0,\ \alpha > 0,\ N_F - \text{const}) \tag{3.16}$$

where $\| F \|$ is the norm of the matrix F.

From the conditions (1.4), (1.5), and (3.16) we conclude that the improper integral on the right-hand side of (3.15) is absolutely convergent, while v_1 is a linear function of the coordinates x_i with bounded coefficients

$$b_k(\eta, t) = 2\int_t^\infty \Big[\sum_{i=1}^n \Big[b_{ij}\sum_{j=1}^n F_{jk}(\tau - t)\Big] \boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) \,/\, \eta, t]\Big] d\tau$$
$$(k = 1, \ldots, n) \tag{3.17}$$

Here, F_{kj} are the elements of the matrix F. These coefficients, on the basis of the conditions (1.5), (1.6), and (3.16), satisfy the inequalities

$$|b_k(\eta, t)| \leqslant 2\int_t^\infty nN_F Bf\,[t, \tau]\, e^{-\alpha(\tau - t)} d\tau < 2\, nBN_F\, f(t) \tag{3.18}$$
$$(B = \|b_{ij}\|, \quad k = 1, \ldots, n)$$

which with (1.7) imply satisfaction of the condition (2.5). Now it is only necessary to prove that the function $v_1(x, \eta, t)$ satisfies Equation (3.13). We calculate the derivative $(d\boldsymbol{M}\{v_1\}/dt_{(3.12)}$ for the function v_1 (3.15). Considering that $x(t)$ in the expression for v_1 is a solution of the system (3.12), and considering that $\eta(t)$ is a random Markov function, we have

$$\left(\frac{d\boldsymbol{M}\{v_1\}}{dt}\right)_{(3.12)}$$
$$= \frac{d}{dt}\, 2\, \boldsymbol{M}\left\{\int_t^\infty \Big[\sum_{i=1}^n \Big(\sum_{j=1}^n b_{ij}\{F(\tau - t)\, x\}_j\Big) \boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) \,/\, \eta(t), t]\Big] d\tau\right\}$$
$$= -2\sum_{i=1}^n \sum_{j=1}^n b_{ij} x_j(t)\, \varphi_i(t, \eta(t)) +$$
$$+ 2\int_t^\infty \Big[\sum_{i=1}^n \Big(\sum_{j=1}^n b_{ij}\frac{\partial}{\partial t}\{F(\tau - t)\, x(t)\}_j\Big) \boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta(t), t]\Big] d\tau +$$
$$+ 2\int_t^\infty \Big[\sum_{i=1}^n \Big(\sum_{j=1}^n b_{ij}\{F(\tau - t)\, x\}_j\Big) \frac{\partial \boldsymbol{M}\,[\boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta(t), t]]}{dt}\Big] d\tau \tag{3.19}$$

The last two terms in the last equation are equal to zero. In fact,

$x(t) = F(t)c$ (c is a constant vector), i.e.

$$\frac{\partial}{\partial t}[F(\tau - t)x(t)] = \frac{\partial}{\partial t}[F(\tau - t)F(t)c] = \frac{\partial}{\partial t}[F(\tau)c] = 0.$$

In the same way

$$\frac{\partial}{\partial t} \boldsymbol{M}\,[\boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta(t), t]]$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{\boldsymbol{M}\,[\boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta(t + \Delta t), t + \Delta t] / \eta(t), t] -$$
$$- \boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta(t), t]\} = 0$$

since the Markovian properties of the function $\eta(t)$ imply that the expression in curl brackets is equal to zero. Consequently, the function v_1 (3.15) indeed satisfies Equation (3.13). This function also obviously satisfies the condition (2.10).

Remark 3.1. Equations (3.15) imply that in order to determine the functions $v_1(x, \eta, t)$ it is sufficient to know the functions

$$\boldsymbol{M}_i(\eta, t, \tau) = \boldsymbol{M}\,[\varphi_i(\tau, \eta(\tau)) / \eta, t] \qquad (i = 1, \ldots, n)$$

i.e. it is sufficient to have, at each time t, the forecast of the future mean values of the perturbations $\phi_i(\tau, \eta(\tau))$, $\tau > t$, on the basis of the knowledge of realized values $\eta(t) = \eta$.

Let us consider Equation (3.11). This equation has the solution $v_0(\eta, t)$ which has the form

$$v_0(\eta, t) = \int_t^\infty \boldsymbol{M}\Big[\Big\{\sum_{i=1}^n [b_i(\eta(\tau), \tau)\,\varphi_i(\tau, \eta(\tau))] -$$
$$- \frac{1}{4}\Big(\sum_{i=1}^n b_i(\eta(\tau), \tau)\, m_i\Big)^2\Big\} / \eta, t\Big] d\tau \tag{3.20}$$

The proof that the function v_0 (3.20) satisfies Equation (3.11), the conditions (2.6) and (2.7), and the second condition (2.11), can be given by the use of the conditions (1.3) to (1.6), similarly to what was done in the case of the function v_1. Therefore, we omit here this proof.

Thus, we arrive at the conclusion that, with the condition 3.1 being satisfied, the functions v and ξ° exist which satisfy the conditions 2.1 to 2.3, i.e. with this condition the problem is solvable. Let us summarize the result obtained.

The problem of construction of an optimum control system $\xi^\circ[x, \eta, t]$ minimizing the mean-squared error

$$J = \int_{t_0}^{\infty} M \left[\sum_{i=1}^{n} x_i^2 (t) + \xi^2 (t)\right] dt$$

in the system (1.1) in the presence of bounded and decreasing random perturbations $\phi_i(t, \eta)$ is solvable if, and only if, the condition 3.1 is satisfied. The optimum control ξ° should be presented in the form

$$\xi^\circ = \xi_1^\circ + \xi_*^\circ (\eta, t) \qquad \left(\xi_1^\circ = \sum_{i=1}^{n} \mu_i x_i\right)$$

The term ξ_1° coincides with the optimum control which is obtained for an analogous system but in the absence of perturbations. The random component $\xi_*^\circ(\eta, t)$ takes into account the existence of random perturbations $\phi_i(t, \eta)$. This term is determined at each instant of time t according to the information on the realized values of $\eta(t)$, but the computational formulas for ξ_*° assume the knowledge of the forecast of the future values of the mean values of the perturbations $\phi_i(r, \eta)$, $r > t$.

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